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Traveling wave solutions of nonlinear evolution equations via the enhanced (G'/G)-expansion method

Kamruzzaman Khan [a], **M. Ali Akbar** [b,*]

[a] *Department of Mathematics, Pabna University of Science and Technology, Bangladesh*
[b] *Department of Applied Mathematics, University of Rajshahi, Bangladesh*

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Abstract In this article, an enhanced (G'/G)-expansion method is suggested to find the traveling wave solutions for the modified Korteweg de-Vries (mKDV) equation. Abundant traveling wave solutions are derived, which are expressed by the hyperbolic and trigonometric functions involving several parameters. The efficiency of this method for finding these exact solutions has been demonstrated. It is shown that the proposed method is effective and can be used for many other nonlinear evolution equations (NLEEs) in mathematical physics.

MATHEMATICS SUBJECT CLASSIFICATION: 35C07; 35C08; 35P99

1. Introduction

Nowadays, NLEEs have been the subject of all-embracing studies in various branches of nonlinear sciences. A special class of analytical solutions named traveling wave solutions for NLEEs has a lot of importance, because most of the phenomena that arise in mathematical physics and engineering fields can be described by NLEEs. NLEEs are frequently used to describe many problems of protein chemistry, chemically reactive materials, in ecology most population models, in physics the heat flow and the wave propagation phenomena, quantum mechanics, fluid mechanics, plasma physics, propagation of shallow water waves, optical fibers, biology, solid state physics, chemical kinematics, geochemistry, meteorology, electricity etc. Therefore, investigating traveling wave solutions is becoming successively attractive in nonlinear sciences day by day. However, not all equations posed of these models are solvable. As a result, many new techniques have been successfully developed by diverse groups of mathematicians and physicists, such as, the Hirota's bilinear transformation method [1,2], the tanh-function method [4,36], the extended tanh-method [5,6], the Exp-function method [7–11], the Adomian decomposition method [12], the F-expansion method [13], the auxiliary equation method [14], the Jacobi elliptic function method [15], Modified Exp-function method [16], the (G'/G)-expansion method [17–26], the Weierstrass elliptic function method [27], the homotopy perturbation method [28–30], the homogeneous balance method [31,32], the modified simple equation method [3,33–35], and so on.

* Corresponding author. Tel.: +880 1918561386; fax: +880 721 750064.
E-mail address: ali_math74@yahoo.com (M. Ali Akbar).
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The objective of this article is to present an enhanced (G'/G)-expansion method for constructing exact solutions for nonlinear evolution equations in mathematical physics via the mKDV equation. The mKdV equation is completely integrable. Therefore, the mKdV equation has N-soliton solutions. The mKDV equation appears in electric circuits and multi-component plasmas.

The article is prepared as follows: In Section 2, the enhanced (G'/G)-expansion method is discussed. In Section 3, we apply this method to the nonlinear evolution equation pointed out above; in Section 4, explanation of the solutions and in Section 5 conclusions are given.

2. The enhanced (G'/G)-expansion method

In this section, we describe the proposed enhanced (G'/G)-expansion method for finding traveling wave solutions of nonlinear evolution equations. Suppose that a nonlinear equation, say in two independent variables x and t is given by

$$\mathcal{R}(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{2.1}$$

where $u(\xi) = u(x, t)$ is an unknown function, $\mathcal{R}$ is a polynomial of $u(x, t)$, and its partial derivatives in which the highest order derivatives and nonlinear terms are involved. In the following, we give the main steps of this method:

Step 1. Combining the independent variables x and t into one variable $\xi = x \pm \omega t$, we suppose that

$$u(\xi) = u(x, t), \quad \xi = x \pm \omega t. \tag{2.2}$$

The traveling wave transformation Eq. (2.2) permits us to reduce Eq. (2.1) to the following ODE:

$$\mathcal{R}(u, u', u'', \ldots) = 0, \tag{2.3}$$

where $\mathcal{R}$ is a polynomial in $u(\xi)$ and its derivatives, while $u'(\xi) = \frac{du}{d\xi}$, $u''(\xi) = \frac{d^2u}{d\xi^2}$, and so on.

Step 2. We suppose that Eq. (2.3) has the formal solution

$$u(\xi) = \sum_{i=-n}^{n} \left(\frac{a_i (G'/G)^i}{(1 + \lambda(G'/G))^i} + b_i (G'/G)^{i-1} \sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)} \right), \tag{2.4}$$

where $G = G(\xi)$ satisfy the equation

$$G'' + \mu G = 0, \tag{2.5}$$

in which $a_i, b_i(-n \leqslant i \leqslant n; n \in \mathbb{N})$ and λ are constants to be determined later, and $\sigma = \pm 1, \mu \neq 0$.

Step 3. The positive integer n can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3). Moreover precisely, we define the degree of $u(\xi)$ as $D(u(\xi)) = n$ which gives rise to the degree of other expression as follows:

$$D\left(\frac{d^q u}{d\xi^q}\right) = n + q, \quad D\left(u^p \left(\frac{d^q u}{d\xi^q}\right)^s\right) = np + s(n + q). \tag{2.6}$$

Therefore, we can find the value of n in Eq. (2.4), using Eq. (2.6).

Step 4. We substitute Eq. (2.4) into Eq. (2.3) using Eq. (2.5) and then collect all terms of same powers of $(G'/G)^j$ and $(G'/G)^j \sqrt{\sigma\left(1 + \frac{1}{\mu}(G'/G)^2\right)}$ together, then set each coefficient of them to zero to yield a over-determined system of algebraic equations, solving this system for $a_i, b_i(-n \leqslant i \leqslant n; n \in \mathbb{N})$, λ and ω.

Step 5. From the general solution of Eq. (2.5), we obtain

When $\mu < 0$,

$$\frac{G'}{G} = \sqrt{-\mu}\ tanh\left(\xi_0 + \sqrt{-\mu}\xi\right). \tag{2.7}$$

And

$$\frac{G'}{G} = \sqrt{-\mu}\ coth\left(\xi_0 + \sqrt{-\mu}\xi\right). \tag{2.8}$$

Again, when $\mu > 0$,

$$\frac{G'}{G} = \sqrt{\mu}\ tan\left(\xi_0 - \sqrt{\mu}\xi\right). \tag{2.9}$$

and

$$\frac{G'}{G} = \sqrt{\mu}\ cot\left(\xi_0 + \sqrt{\mu}\xi\right). \tag{2.10}$$

where ξ_0 is an arbitrary constant. Finally, substituting $a_i, b_i(-n \leqslant i \leqslant n; n \in \mathbb{N})$, λ, ω and Eqs. (2.7)–(2.10) into Eq. (2.4), we obtain traveling wave solutions of Eq. (2.1).

3. Application

In this section, we will exert enhanced (G'/G)-expansion method to solve the mKDV equation in the form

$$u_t - u^2 u_x + \delta u_{xxx} = 0, \tag{3.1}$$

where δ is a nonzero constant.

The traveling wave transformation equation $u(\xi) = u(x, t)$, $\xi = x \pm \omega t$ transform Eq. (3.1) to the following ordinary differential equation:

$$-\omega u' - u^2 u' + \delta u''' = 0. \tag{3.2}$$

Integrating Eq. (3.2) with respect to ξ once, we obtain

$$\delta u'' - \omega u - \frac{u^3}{3} + C = 0, \tag{3.3}$$

where C is an integration constant. Balancing the highest-order derivative term u'' and the nonlinear term of the highest order u^3, yields $3n = n + 2$, which gives $n = 1$.

Hence for $n = 1$, Eq. (2.4) reduces to

$$u(\xi) = a_0 + \frac{a_1(G'/G)}{1 + \lambda(G'/G)} + \frac{a_{-1}(1 + \lambda(G'/G))}{(G'/G)} + b_0 (G'/G)^{-1} \sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)} + b_1 \sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)} + b_{-1}(G'/G)^{-2} \sqrt{\sigma\left(1 + \frac{(G'/G)^2}{\mu}\right)}, \tag{3.4}$$

where $G = G(\xi)$ satisfies Eq. (2.5). Substituting Eq. (3.4) with Eq. (2.5) into Eq. (3.3), we obtain a polynomial of $(G'/G)^j$ and

$(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$. From this polynomial, we equate the coefficients of $(G'/G)^j$ and $(G'/G)^j\sqrt{\sigma\left(1+\frac{(G'/G)^2}{\mu}\right)}$. Setting them to zero, we obtain a over-determined system that consists of twenty-five algebraic equations (for minimalism, we omitted to display them). Solving these systems of algebraic equation, we obtain

Set 1: $C=0,\ \omega=-4\delta\mu,\ \lambda=0,\ a_{-1}=\pm\mu\sqrt{6\delta},\ a_0=0,$ $a_1==\pm\sqrt{6\delta},\ b_1=b_0=b_1=0.$

Set 2: $C=0,\ \omega=2\delta\mu,\ \lambda=0,\ a_{-1}=0,\ a_0=0,\ a_1==\pm\sqrt{6\delta},\ b_1=b_0=b_1=0.$

$C=0,\ \omega=2\delta\mu,\ \lambda=\lambda,\ a_{-1}=\pm\mu\sqrt{6\delta},\ a_0=\mp\mu\lambda\sqrt{6\delta},\ a_1=b_{-1}=b_0=b_1=0.$

Set 3: $C=0,\ \omega=\frac{1}{2}\delta\mu,\ \lambda=0,\ a_{-1}=0,\ a_0=0,\ a_1=\pm\sqrt{\frac{3\delta}{2}},$ $b_{-1}=0,\ b_0=0,\ b_1=\pm\sqrt{\frac{3\delta\mu}{2\sigma}}.$

Set 4: $C=\mp\frac{24\delta^2\mu^2(\lambda^2\mu+1)\lambda}{\sqrt{6\delta}},\ \omega=-2\delta\mu(2+3\lambda^2\mu),\ \lambda=\lambda,$ - $a_{-1}=\pm\mu\sqrt{6\delta},\ a_0=\mp\mu\lambda\sqrt{6\delta},$

$a_1=\pm\sqrt{6\delta}(\lambda^2\mu+1),\ b_{-1}=b_0=b_1=0.$

Set 5: $C=0,\ \omega=-\delta\mu,\ \lambda=\lambda,\ a_{-1}=a_0=a_1=b_{-1}=b_0=0,$ $b_1=\pm\sqrt{\frac{6\delta\mu}{\sigma}}.$

$C=0,\ \omega=-\delta\mu,\ \lambda=\lambda,\ a_{-1}=a_0=a_1=b_{-1}=0,\ b_0=\pm\mu\sqrt{\frac{6\delta}{\sigma}},\ b_1=0.$

Set 6: $C=0,\omega=2\delta\mu,\ \lambda=\lambda,\ a_{-1}=0,\ a_0=\mp\mu\lambda\sqrt{6\delta},$ $a_1=\pm\sqrt{6\delta}(\lambda^2\mu+1),\ b_{-1}=b_0=b_1=0.$

Set 7: $C=0,\ \omega=\frac{1}{2}\delta\mu,\ \lambda=\lambda,\ a_{-1}=\pm\mu\sqrt{\left(\frac{3\delta}{2}\right)},$ $a_0=\mp\lambda\sqrt{\left(\frac{3\delta}{2}\right)},$

$a_1=b_{-1}=b_1=0,\ b_0=\pm\mu\sqrt{\left(\frac{3\delta}{2\sigma}\right)}.$

Now substituting Sets 1–7 with Eq. (2.5) into Eq. (3.4), we obtain abundant traveling wave solutions of Eq. (3.1) as follows:

When $\mu<0$, we obtain the following hyperbolic solutions:

Family 1:

$$u_1(\xi)=\pm\sqrt{(-6\delta\mu)}\left(coth\left(\xi_0+\sqrt{-\mu}\xi\right)-tanh\left(\xi_0+\sqrt{-\mu}\xi\right)\right)$$

where $\xi=x+4\delta\mu t.$

Family 2:

$$u_2(\xi)=\pm\sqrt{-6\delta\mu}\ tanh(\xi_0+\sqrt{-\mu}\xi).$$

$$u_3(\xi)=\pm\sqrt{-6\delta\mu}\ coth(\xi_0+\sqrt{-\mu}\xi).$$

where $\xi=x-2\delta\mu t.$

Family 3:

$$u_4(\xi)=\pm\sqrt{\left(\frac{3\delta}{2}\right)}\left(\sqrt{-\mu}\ tanh\left(\xi_0+\sqrt{-\mu}\xi\right)\right.$$
$$\left.\pm\sqrt{\mu}\ sech\ \left(\xi_0+\sqrt{-\mu}\xi\right)\right).$$

$$u_5(\xi)=\pm\sqrt{\left(\frac{3\delta}{2}\right)}\left(\sqrt{-\mu}\ coth\left(\xi_0+\sqrt{-\mu}\xi\right)\right.$$
$$\left.\pm I\sqrt{\mu}\ cosech\ \left(\xi_0+\sqrt{-\mu}\xi\right)\right).$$

where $\xi=x-\frac{1}{2}\delta\mu t.$

Family 4:

$$u_6(\xi)=\mp\sqrt{-6\delta\mu}\left(\frac{\lambda\sqrt{-\mu}-(\mu\lambda^2+1)tanh\left(\xi_0+\sqrt{-\mu}\xi\right)+coth\left(\xi_0+\sqrt{-\mu}\xi\right)}{1+\lambda\sqrt{-\mu}tanh\left(\xi_0+\sqrt{-\mu}\xi\right)}\right).$$

$$u_7(\xi)=\mp\sqrt{-6\delta\mu}\left(\frac{\lambda\sqrt{-\mu}-(\mu\lambda^2+1)coth(\xi_0+\sqrt{-\mu}\xi)+tanh(\xi_0+\sqrt{-\mu}\xi)}{1+\lambda\sqrt{-\mu}coth(\xi_0+\sqrt{-\mu}\xi)}\right).$$

where $\xi=x+2\delta\mu(2+3\lambda^2\mu)t.$

Family 5:

$$u_8(\xi)=\pm\sqrt{6\delta\mu}\ sech(\xi_0+\sqrt{-\mu}\xi).$$

$$u_9(\xi)=\pm I\sqrt{6\delta\mu}\ cosech(\xi_0+\sqrt{-\mu}\xi).$$

where $\xi=x+\delta\mu t.$

Family 6:

$$u_{10}(\xi)=\pm\sqrt{6\delta}\left(\frac{-\mu\lambda+\sqrt{-\mu}tanh(\xi_0+\sqrt{-\mu}\xi)}{1+\lambda\sqrt{-\mu}tanh(\xi_0+\sqrt{-\mu}\xi)}\right).$$

$$u_{11}(\xi)=\pm\sqrt{6\delta}\left(\frac{-\mu\lambda+\sqrt{-\mu}coth(\xi_0+\sqrt{-\mu}\xi)}{1+\lambda\sqrt{-\mu}coth(\xi_0+\sqrt{-\mu}\xi)}\right).$$

where $\xi=x-2\delta\mu t$

Family 7:

$$u_{12}(\xi)=\pm\sqrt{\left(\frac{3\delta}{2}\right)}\left(coth\left(\xi_0+\sqrt{-\mu}\xi\right)\pm cosech\left(\xi_0+\sqrt{-\mu}\xi\right)\right).$$

$$u_{13}(\xi)=\pm\sqrt{\left(\frac{3\delta}{2}\right)}\left(tanh\left(\xi_0+\sqrt{-\mu}\xi\right)\pm Isech\left(\xi_0+\sqrt{-\mu}\xi\right)\right).$$

where $\xi=x-\frac{1}{2}\delta\mu t$.Again, when $\mu>0$, we obtain the following plane periodic solutions:

Family 8:

$$u_{14}(\xi)=\pm\sqrt{(6\delta\mu)}\left(cot\left(\xi_0\pm\sqrt{\mu}\xi\right)+tan\left(\xi_0\pm\sqrt{\mu}\xi\right)\right).$$

where $\xi=x+4\delta\mu t.$

Family 9:

$$u_{15}(\xi)=\pm\sqrt{6\delta\mu}tan(\xi_0\mp\sqrt{\mu}\xi).$$

$$u_{16}(\xi)=\pm\sqrt{6\delta\mu}cot(\xi_0\pm\sqrt{\mu}\xi).$$

where $\xi=x-2\delta\mu t.$

Family 10:

$$u_{17}(\xi)=\pm\sqrt{\left(\frac{3\delta\mu}{2}\right)}\left(tan\left(\xi_0-\sqrt{\mu}\xi\right)\pm\sec\left(\xi_0-\sqrt{\mu}\xi\right)\right).$$

$$u_{18}(\xi)=\pm\sqrt{\left(\frac{3\delta\mu}{2}\right)}\left(cot\left(\xi_0+\sqrt{\mu}\xi\right)\pm cosec\left(\xi_0+\sqrt{\mu}\xi\right)\right).$$

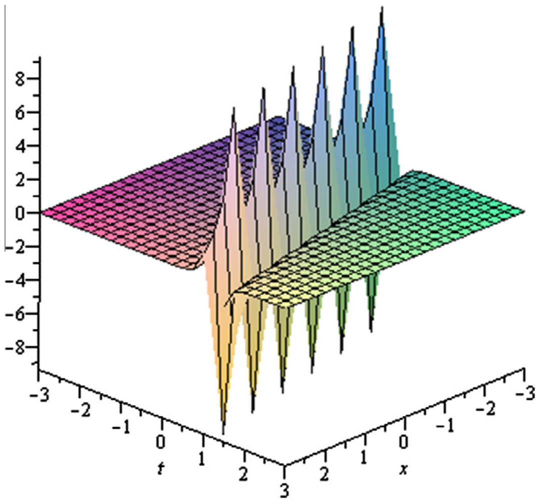

Figure 1 Family 1 ($u_1(\xi)$), $\delta = 1$, $\xi_0 = 2$, $-3 \leqslant x$, $t \leqslant 3$.

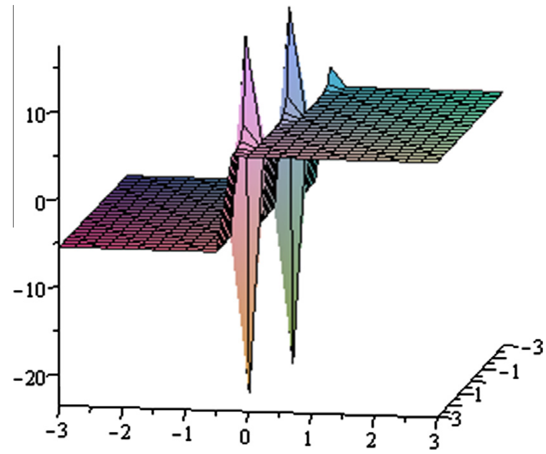

Figure 4 Family -2 ($u_3(\xi)$): $\delta = 1$, $\mu = -5$, $\xi_0 = 0$, $-3 \leqslant x$, $t \leqslant 3$.

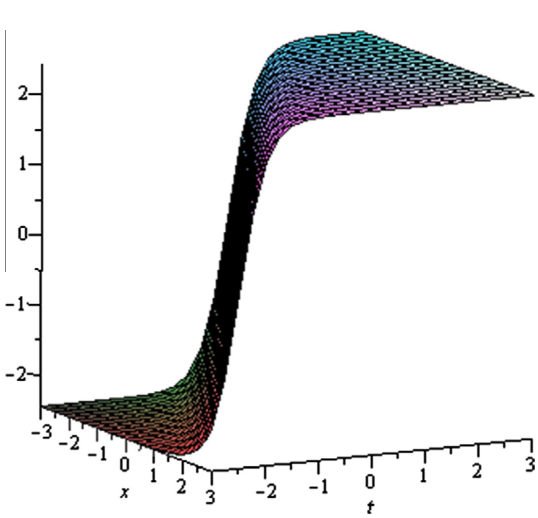

Figure 2 Family 2($u_2(\xi)$): $\delta = 1$, $\xi_0 = 2$, $-3 \leqslant x$, $t \leqslant 3$.

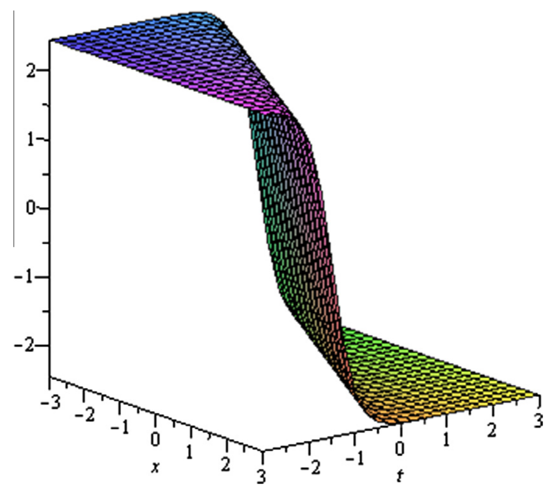

Figure 5 Family -4: $u_7(\xi)$, $\delta = 1$, $\mu = -1$, $\lambda = 1$, $\xi_0 = 0$, $-3 \leqslant x$, $t \leqslant 3$.

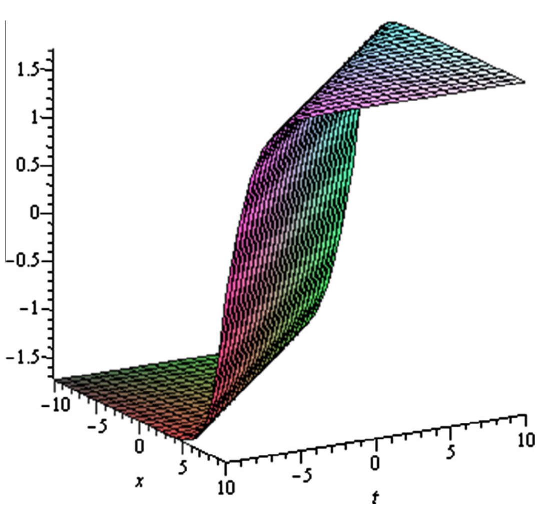

Figure 3 Family -3: $u_5(\xi)$, $\delta = 1$, $\mu = -5$, $\xi_0 = 0$, $-10 \leqslant x$, $t \leqslant 10$.

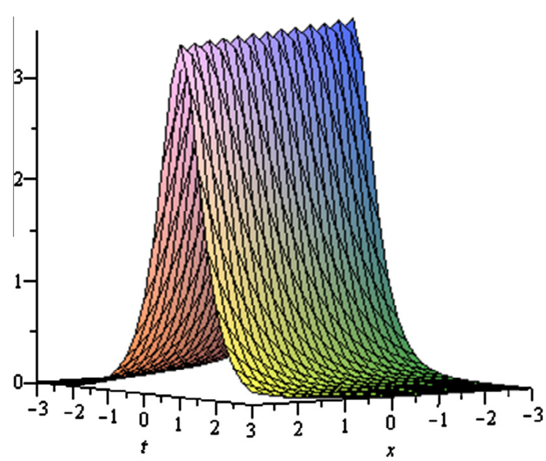

Figure 6 Family -5: $u_8(\xi)$, $\delta = 2$, $\mu = -1$, $\lambda = 2$, $\xi_0 = -1$, $-3 \leqslant x$, $t \leqslant 3$.

where $\xi = x - \frac{1}{2}\delta\mu t$.

Family 11:

$$u_{19}(\xi) = \pm\sqrt{6\delta\mu}\left(\frac{(\mu\lambda^2+1)tan(\xi_0 - \sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}tan(\xi_0 - \sqrt{\mu}\xi)} - cot(\xi_0 - \sqrt{\mu}\xi)\right).$$

$$u_{20}(\xi) = \pm\sqrt{6\delta\mu}\left(\frac{(\mu\lambda^2+1)cot(\xi_0 + \sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}cot(\xi_0 + \sqrt{\mu}\xi)} + tan(\xi_0 + \sqrt{\mu}\xi)\right)$$

where $\xi = x + 2\delta\mu(2 + 3\lambda^2\mu)t$.

Family 12:

$$u_{21}(\xi) = \pm\sqrt{6\delta\mu}\ sec(\xi_0 \mp \sqrt{\mu}\xi).$$

$$u_{22}(\xi) = \pm\sqrt{6\delta\mu}\ cosec(\xi_0 \pm \sqrt{\mu}\xi).$$

where $\xi = x + \delta\mu t$.

Family 13:

$$u_{23}(\xi) = \pm\sqrt{6\delta}\left(\frac{-\mu\lambda + \sqrt{\mu}tan(\xi_0 - \sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}tan(\xi_0 - \sqrt{\mu}\xi)}\right).$$

$$u_{24}(\xi) = \pm\sqrt{6\delta}\left(\frac{-\mu\lambda + \sqrt{\mu}cot(\xi_0 + \sqrt{\mu}\xi)}{1+\lambda\sqrt{\mu}cot(\xi_0 + \sqrt{\mu}\xi)}\right).$$

where $\xi = x - 2\delta\mu t$

Family 14:

$$u_{25}(\xi) = \pm\sqrt{\left(\frac{3\delta\mu}{2}\right)}\left(cot\left(\xi_0 - \sqrt{\mu}\xi\right) \pm cosec\left(\xi_0 - \sqrt{\mu}\xi\right)\right).$$

$$u_{26}(\xi) = \pm\sqrt{\left(\frac{3\delta\mu}{2}\right)}\left(tan\left(\xi_0 + \sqrt{\mu}\xi\right) \pm sec\left(\xi_0 + \sqrt{\mu}\xi\right)\right).$$

where $\xi = x - \frac{1}{2}\delta\mu t$.

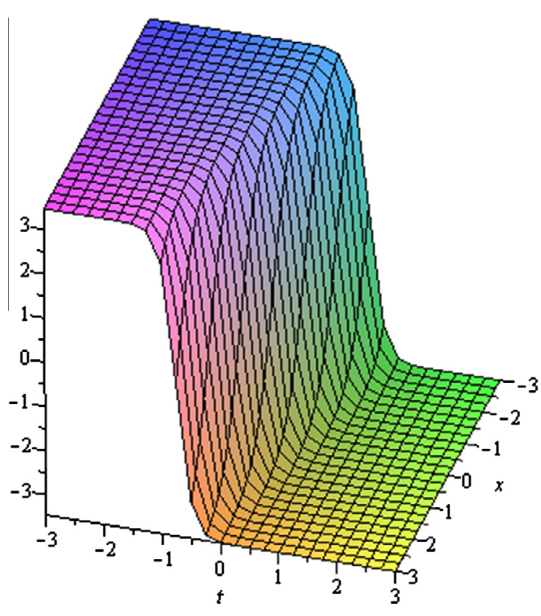

Figure 7 Family –6: $u_{10}(\xi)$, $\delta = 2$, $\mu = -1$, $\lambda = 0$, $\xi_0 = 0$, $-3 \leqslant x, t \leqslant 3$.

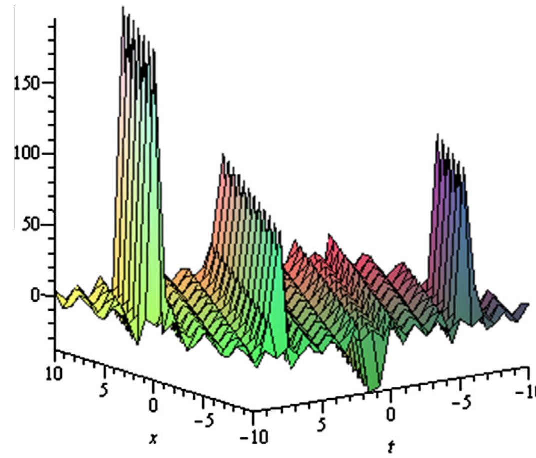

Figure 8 Family –8: $u_{14}(\xi)$, $\delta = 0.5$, $\mu = 1$, $\xi_0 = 2$, $-10 \leqslant x, t \leqslant 10$.

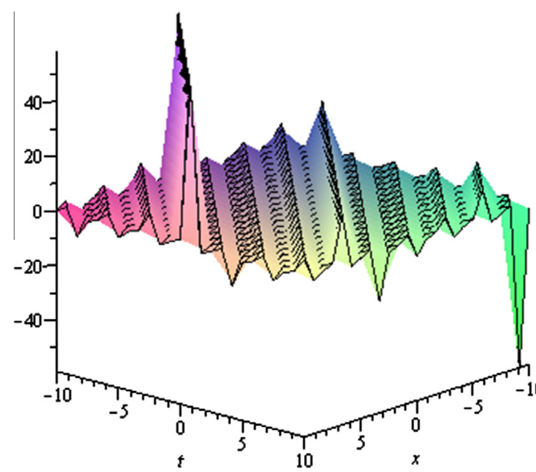

Figure 9 Family –9 $u_{15}(\xi)$, $\delta = 5$, $\mu = 1$, $\xi_0 = 5$, $-10 \leqslant x, t \leqslant 10$.

4. Explanation of the solutions and graphical representations

4.1. Explanation of the solutions

In this sub-section, we will discuss the explanation of obtained solutions of the mKDV equation. It is interesting to point out that the delicate balance between the nonlinearity effect of u^2u_x and the dispersion effect of u_{xxx} gives rise to solitons, that after a fully interaction with others, the solitons come back retaining their identities with the same speed and shape. The mKdV equation has solitary wave solutions that have exponentially decaying wings. If two solitons of the mKdV equation collide, the solitons just pass through each other and emerge unchanged.

The determined solutions from Family 1 to 7 for $\mu < 0$ are hyperbolic function solutions, said to be travelling wave solutions. For $\mu < 0$, Family 1–3 ($u_5(x, t)$), Family 4–5 ($u_9(x, t)$), Family 6–7 ($u_{12}(x, t)$) all are soliton like solutions which are special kind of solitary waves. For $\mu < 0$, Family-3 ($u_4(x, t)$), Family-5($u_8(x, t)$), and Family-7 ($u_{13}(x, t)$) are complex

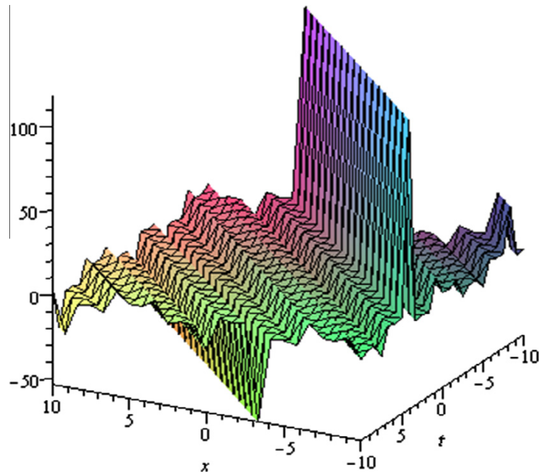

Figure 10 Family −12: $u_{21}(\xi)$, $\delta = 5$, $\mu = 1$, $\xi_0 = 5$, $-10 \leqslant x, t \leqslant 10$.

soliton solutions. Family-2 ($u_2(x, t)$) gives kink solutions and Family-2 ($u_3(x, t)$) gives singular kink wave solutions.

Again for $\mu > 0$, Family 8-14 are trigonometric function solutions, also said to be traveling wave solutions that are periodic. The wave speed ω plays an important role in the physical structure of the solutions obtained above. For the positive values of wave speed ω, the disturbance represented by $u(\xi) = u(x - \omega t)$ is moving in the positive x-direction. Consequently, the negative values of wave speed ω, the disturbances represented by $u(\xi) = u(x - \omega t)$ are moving in the negative x–direction. Moreover, the graphical demonstrations of some obtained solutions for particular values of the parameters are shown in Figs. 1–10 in the following sub-section.

4.2. Graphical representations

Some of our obtained traveling wave solutions are represented in the following figures with the aid of commercial software Maple:

5. Conclusion

In this article, we have established an enhanced (G'/G)-expansion method and utilized it to find the exact solutions of nonlinear equations with the help of symbolic computation software Maple. We have successfully obtained abundant traveling wave solutions of the mKDV equation. When the parameters are taken as special values, the solitary wave solutions and the periodic wave solutions have originated from the exact solutions. Taken as a whole, it is worthwhile to mention that this method is effective for solving other nonlinear evolution equations in mathematical physics.

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